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Dissipative mass-accreting quantum oscillator

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Abstract. We begin by revisiting the so-called Caldirola–Kanai Hamiltonian and discuss its inherent ambiguity: does it represent a dissipative harmonic oscillator (HO) subject to a friction force, or does it describe an HO with a time-dependent mass (TDM)? Although classically both descriptions do coexist, in the quantum domain the solution of the Schrödinger equation (or Heisenberg equations of motion) with a TDM does not present inconsistencies, however, the dissipative Hamiltonian shows violation of the Heisenberg uncertainty principle. This violation is avoided by introducing a stochastic force in the equations of motion, which will take care of the fluctuations due to the environment. Once the distinction between the dissipative and amplifying Hamiltonian is made clear, we consider the problem of the quantum TDM HO subject to dissipation, showing that both phenomena may be merged and described by a single Hamiltonian, the *amplifying–dissipative Hamiltonian*. We obtain the solutions of the Heisenberg equations of motion for the canonical momentum and position; next, we specialize on the weak damping limit and analyse the effects of the amplifying–dissipative process on the mean values of the physical variables.

1. Introduction

From the phenomenological point of view, the problem of macroscopic friction is satisfactorily described by Newtonian mechanics [1, 2], since suitable friction forces can always be added to the equations of motion, although the task of solving the differential equations remains. As friction forces are not conservative, they cannot be included in a Lagrange function without introducing some ambiguity [3, 4]. Nevertheless, physicists have been persistent in trying to introduce that phenomenon into the classical frameworks of Lagrange and Hamilton [3–6]. This interest in describing the friction by Lagrangian and Hamiltonian mechanics grew with the arrival of quantum mechanics, because the new theory was essentially a Hamiltonian one, and researchers were and are always looking for the quantum version of a classical motion.

Since the pioneer work of Bateman [3] on variational principles related to the Lagrangian formalism of non-conservative systems, the paradigm to the problem of friction has been the harmonic oscillator (HO) subject to a force proportional to the velocity. Later, Caldirola [7] and Kanai [8] introduced, independently, a Hamiltonian function, that is known after their name, the CK Hamiltonian, see (3), which is the simplest one that permits Newton's equation of motion to be derived,

$$\ddot{q} + (\gamma/m)\dot{q} + \omega_0^2 q = 0. \tag{1}$$

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Thus, with a classical Hamiltonian describing friction, the quantization becomes possible by using the usual prescription of substituting c-numbers by q-numbers, but several inconsistencies arise, the most serious being the violation of the Heisenberg uncertainty principle. For more details see the review paper of Dekker [9]. Nevertheless, many papers continued to be written on this theme, containing new insights, solutions and proposals in order to surmount the conceptual and formal difficulties, because that problem (a special case of the general class of TD quadratic Hamiltonians) is useful for understanding the physics of open systems [10–14].

In order to remove the uncertainty principle violation, the CK Hamiltonian (3) received another interpretation [15–17]: Newton's equation (1) can also describe a distinct phenomenological problem, namely, an HO, not subject to a friction force, but containing a time-dependent mass (TDM) with exponential accretion, $m(t) = m_0 \exp(\zeta t)$, where $\zeta = \gamma/m$. Thus, one has one single second-order differential equation, (1), representing two quite different problems, with, naturally, only one solution. However, as will be seen below, this ambiguity can be removed by conveniently defining the meaning of the Hamiltonian and its variables. It is worth noting that from the classical point of view both interpretations are physically possible, the CK Hamiltonian describing either a dissipative HO or a TDM HO, with no inconsistencies in the analysis of the solutions.

According to Havas [4], dissipative systems can be handled following this procedure: first, the Lagrange function is multiplied by an adequate *integrating factor* and second, the canonical momentum and position must be distinguished from the physical ones, where coincidence only occurs for non-dissipative systems. Thus, following this recipe, the Hamiltonian function no longer represents the energy of the system, but it continues to be the generator of its motion; the total energy, as usual, is written as the sum of the kinetic plus potential energies. For the TDM HO nothing changes, the Hamiltonian and energy continue to coincide, and no distinction arises between the canonical and physical variables (position and linear momentum).

The quantization of the TDM HO does not present any dispute about its interpretation or consistency, however, because the quantized dissipative HO the introduction of the integrating factor, *per se*, in the Lagrange function is not sufficient to guarantee the Heisenberg uncertainty inequality. Nevertheless, this flaw can be remedied by adding, *ad hoc*, to the conservative force, a phenomenological time-dependent (TD) stochastic force [18, 19]. Borrowed from the Langevin equation for the Brownian motion, this procedure avoids the occurrence of unphysical results, such as the HO mean energy going asymptotically to a value below the ground state, or the violation of the Heisenberg or Robertson-Schrödinger uncertainty inequalities. Therefore, both the integrating factor plus the TD stochastic force will respond to the effects of an environment acting on the system, causing energy dissipation or absorption until thermal equilibrium is reached. In order to verify the consistency of this procedure it was shown [19] that the introduction of these two elements led to the verification of the fluctuation–dissipation theorem [20].

The microscopic origin of the CK Hamiltonian was originally demonstrated in [9]: a transformed Hamiltonian of the whole system consisting of a central HO interacting with a set of N ($N \gg 1$) HOs (the *bath* or *reservoir*) can be written as the sum of two commuting terms, the first one is exactly the CK Hamiltonian whereas the second is related to the reservoir. Later, the authors of [21] related the CK Hamiltonian to the problem of Caldeira and Leggett [22], showing that the wavefunction of the whole system (without the use of path-integral formalism) is factorized as a direct product to two wavefunctions associated with two independent Hilbert spaces. The wavefunction of the central HO is the solution of the Schrödinger equation with CK Hamiltonian, (although also involving the coordinates of the bath), whereas the other is a direct product of the wavefunctions of the N oscillators. Thus, although the whole wavefunction is factorizable into two wavefunctions, the coordinates of the central HO and bath oscillators cannot be disentangled. This implies that the CK Hamiltonian alone is not sufficient to respond to the quantum dissipative problem (as already noted in [9]) therefore the presence of the bath is essential for a correct treatment. Here we substitute the effect of the bath by an effective Markovian TD stochastic force.

In this work we are concerned with the problem of the concomitant mass increment (TDM HO) and energy dissipation (HO interacting with the environment). Thus, we introduce the *amplifier–dissipative* (AD) Hamiltonian and also distinguish between the canonical or mathematical variables and the physical ones. Subsequently, the AD Hamiltonian is quantized, the Heisenberg equations of motion are solved and the mean values of observables compared with those occurring in two particular situations, the TDM HO and the purely dissipative HO. From the general point of view the problem treated here is quite realistic, for instance, an electromagnetic mode being pumped in a cavity and suffering dissipation on its walls; or trapped atoms interacting with their environment or radiation [23]. The present treatment of dissipation was used for a particle trapped by oscillating fields, when a time-dependent frequency occurs [24]. The friction force arises as a result of the viscosity created by the presence of a background gas constantly colliding with the trapped particle or it may be the result of photon interaction in the so-called optical molasses [25].

The paper is organized as follows: in section 2 the conceptual differences between a purely dissipative HO and a non-dissipative TDM HO are outlined. In section 3 we consider the TDM HO subject to dissipation and show that the usual integrating factor, introduced to take care of pure dissipation, is modified as a result of to the presence of the TDM; next we obtain the classical and quantum AD Hamiltonians. In section 4 the Heisenberg equations of motion for the canonical variables are solved exactly. In section 5, we discuss the solutions for the under-amplified case in the weak friction limit, and we also recover old solutions for the dissipative case with time-independent mass. Section 6 is devoted to a summary and conclusions, whereas the appendix contains the details of the solution of the Heisenberg equations of motion.

2. Revisiting the Caldirola-Kanai Hamiltonian: the dissipative versus time-dependent mass harmonic oscillator

We briefly review the CK Hamiltonian that historically [7, 8] describes the phenomenon of energy dissipation. We then compare the expressions for the energy, position and momentum variables, with those obtained from the same Hamiltonian although adopting the TDM HO interpretation, which *is not* dissipative. In this section the treatment is classical.

The usual way to introduce friction in Newtonian mechanics is done by adding, in Newtons's second law, a force proportional to the velocity, $-\gamma\dot{q}$. However, there is an ambiguity in defining the Lagrange function from which one could derive the equations of motion; such an ambiguity was thoroughly studied in [4] and the existence of several Lagrange functions (equivalent Lagrangians), or more exactly, several integrating factors, (leading to the same equations of motion) was noticed, although leading to different Hamiltonians. The simplest Lagrange function is

$$L(q, \dot{q}, t) = \left[\tfrac{1}{2}m\dot{q}^2 - V(q)\right]\mathrm{e}^{\lambda t} \tag{2}$$

where the exponential factor is the *integrating factor*. By using Lagrange equations, Newton's second law, (1), is recovered. The Hamilton function is obtained following the usual recipe of the theory [1, 2], and for a harmonic force, it is

$$H(p, q, t) = \frac{p^2}{2m}\,\mathrm{e}^{-\lambda t} + \frac{1}{2}m\omega_0^2 q^2\,\mathrm{e}^{\lambda t} \tag{3}$$

called the CK Hamiltonian, after the pioneer works of Caldirola and Kanai [7, 8]. However, care must be taken when defining the physical quantities. With the introduction of the integrating factor, the HO is not a closed system, thus (3) no longer stands for the energy, but continues as the generator of the motion. We must also distinguish between canonical momentum p and the physical momentum p_{phys},

$$p = \frac{\partial L}{\partial \dot{q}} = m\dot{q}\,\mathrm{e}^{\lambda t} \qquad p_{\mathrm{phys}} = m\dot{q} = p\,\mathrm{e}^{-\lambda t}. \tag{4}$$

The expressions for the position and physical momentum are

$$q(t) = [a\,\mathrm{e}^{\mathrm{i}\omega t} + a^*\,\mathrm{e}^{-\mathrm{i}\omega t}]\,\mathrm{e}^{-\frac{1}{2}\lambda t} \tag{5}$$

$$p_{\mathrm{phys}}(t) = m\left[a\left(-\frac{\lambda}{2} + \mathrm{i}\omega\right)\mathrm{e}^{\mathrm{i}\omega t} - a^*\left(\frac{\lambda}{2} + \mathrm{i}\omega\right)\mathrm{e}^{-\mathrm{i}\omega_0 t}\right]\mathrm{e}^{-\frac{1}{2}\lambda t} \tag{6}$$

where a and a^* are constants determined from the initial conditions p_0 and q_0,

$$a = \frac{p_0 + mq_0(\frac{1}{2}\lambda + \mathrm{i}\omega)}{2\mathrm{i}m\omega} \tag{7}$$

and the shifted frequency $\omega = [\omega_0^2 - (\lambda/2)^2]^{1/2}$ is assumed to be real and corresponds to the under-damped regime. Writing the total energy as the sum of the kinetic plus potential energies, $E = \frac{1}{2}m\dot{q}^2 + \frac{1}{2}m\omega_0^2 q^2$, we obtain the following relation between the energy and Hamiltonian function,

$$E = H(p, q, t)\,\mathrm{e}^{-\lambda t}. \tag{8}$$

As $t \to \infty$, the position, physical momentum and energy go eventually to zero.

Now, if one considers the HO having a TDM with exponential accretion, $m(t) = m_0\,\mathrm{e}^{\zeta t}$, where $\zeta \equiv \lambda$, the equation of motion is the same as (1). However, adequately defining the momentum (now that the canonical and physical coincide) and energy (which coincides with the Hamiltonian function), one verifies that there is no dissipation as in the previous consideration: the Lagrange function is

$$L'(q, \dot{q}, t) = \tfrac{1}{2}m(t)\dot{q}^2 - \tfrac{1}{2}m(t)\omega_0^2 q^2 \tag{9}$$

and the Hamiltonian function is

$$H(p, q, t) = E = \frac{p^2}{2m(t)} + \frac{1}{2}m(t)\omega_0^2 q^2. \tag{10}$$

The expression for the position is

$$q(t) = [a\,\mathrm{e}^{\mathrm{i}\Omega_0 t} + a^*\,\mathrm{e}^{-\mathrm{i}\Omega_0 t}]\,\mathrm{e}^{-\frac{1}{2}\zeta t} \tag{11}$$

which coincides with (5) and the linear momentum is

$$p(t) = m_0\left[a\left(-\frac{\zeta}{2} + \mathrm{i}\Omega_0\right)\mathrm{e}^{\mathrm{i}\Omega_0 t} - a^*\left(\frac{\zeta}{2} + \mathrm{i}\Omega_0\right)\mathrm{e}^{-\mathrm{i}\Omega_0 t}\right]\mathrm{e}^{\frac{1}{2}\zeta t} \tag{12}$$

that differs from (6) by the sign in the argument of the exponential factor, and $\Omega_0^2 = \omega_0^2 - (\zeta/2)^2$. So, we see that for $\zeta > 0$, as $t \to \infty$, the position goes to 0, the momentum goes to ∞ (in quantum mechanics this behaviour leads to the phenomenon of squeezing) whereas the energy remains finite but not constant,

$$E = H(p, q, t) = \frac{1}{2}m_0\left[a\left(-\frac{\zeta}{2} + \mathrm{i}\Omega_0\right)\mathrm{e}^{\mathrm{i}\Omega_0 t} - a^*\left(\frac{\zeta}{2} + \mathrm{i}\Omega_0\right)\mathrm{e}^{-\mathrm{i}\Omega_0 t}\right]^2 + \frac{1}{2}m_0\omega_0^2\left[a\,\mathrm{e}^{\mathrm{i}\Omega_0 t} + a^*\,\mathrm{e}^{-\mathrm{i}\Omega_0 t}\right]^2. \tag{13}$$

Taking the average of equation (13) over one period of oscillation results in

$$\overline{E}^T = \frac{\omega_0^2}{\Omega_0^2}\left[\frac{1}{2}m\omega_0^2 q_0^2 + \frac{1}{2}\frac{p_0^2}{m} + \frac{1}{2}\zeta p_0 q_0\right] = \frac{\omega_0^2}{\Omega_0^2}\left[E(0) + \frac{1}{2}\zeta p_0 q_0\right] \tag{14}$$

which, for conveniently chosen initial conditions and $\zeta/2 \simeq \omega_0$, can be much larger than the energy (constant of motion) for the case of constant mass, $E(0) = \frac{1}{2}(m\omega_0^2 q_0^2 + p_0^2/m)$, thus characterizing the *energy amplification*.

Therefore, we saw that a unique equation of motion leads to two conceptually different physical problems. In the first case the energy is dissipated while in second it is amplified. Now we shall look at the situation when both phenomena, amplification and dissipation, occur simultaneously.

3. The phenomenological AD Hamiltonian

As in the previous section, the effect of friction is introduced in the Lagrange function as an integrating factor, however, the argument in the exponential factor can no longer be considered as linear in t,

$$L(q, \dot{q}; t) = [\tfrac{1}{2}m(t)\dot{q}^2 - V(q; t) + qF(t)]\,\mathrm{e}^{\gamma(t)} \tag{15}$$

the function $\gamma(t)$ is to be determined such that the equation of motion, derived from Lagrangian formalism, must be the same as that written from Newtonian equations. The additional potential $qF(t)$, where $F(t)$ is a real stochastic force, is introduced in order to take care of fluctuations of the physical observables. The integrating factor and the stochastic potential both respond to the problem of friction, produced by the action of the environment on the HO, for the description of the Brownian particle. In quantum mechanics the introduction of a stochastic force is crucial in order to avoid the violation of the Heisenberg uncertainty inequality.

The second-order differential equation, for the position q, obtained from the Lagrange equations and from Newton's second law are,

$$\ddot{q} = -\left(\frac{\dot{m}}{m} + \dot{\gamma}\right)\dot{q} + \frac{1}{m}\left[-\frac{\partial V(q;t)}{\partial q} + F(t)\right] \tag{16}$$

$$\ddot{q} = -\left(\frac{\dot{m}}{m} + \frac{\lambda}{m}\right)\dot{q} + \frac{1}{m}\left[-\frac{\partial V(q;t)}{\partial q} + F(t)\right] \tag{17}$$

respectively, and $m = m_0\,\mathrm{e}^{\zeta t}$ with $\zeta \geqslant 0$. The above equations coincide for

$$\gamma(t) = \frac{\gamma_0}{\zeta}(1 - \mathrm{e}^{-\zeta t}) \tag{18}$$

where $\gamma_0 = \lambda/m_0$. The following limits hold,

$$\lim_{t\to\infty}\gamma(t) = \frac{\gamma_0}{\zeta} \qquad \text{and} \qquad \lim_{\zeta\to 0}\gamma(t) = \gamma_0 t \tag{19}$$

where the second limit recovers the argument in the exponential factor for the purely dissipative Lagrange function. Here we shall consider that the stochastic force is Markovian, its mean in a statistical ensemble is zero and its autocorrelation function is a Dirac delta-function (memoryless): $\langle F(t)\rangle = 0$ and $\langle F(t)F(t')\rangle = 2d\delta(t - t')$. The constant d is determined such as to satisfy the fluctuation–dissipation theorem [18, 19, 26, 27]

$$d = \frac{\hbar}{2} m_0 \omega_0 \gamma_0 \frac{\Omega_0^2}{\omega_0^2} \coth\left(\frac{\hbar\omega_0}{2k_B T}\right) \tag{20}$$

and the prefactor Ω_0^2/ω_0^2 avoids the unphysical asymptotic expressions introduced in section 5. In the present semi-phenomenological context we cannot determine whether the argument of 'coth' is modified by the TDM, only a microscopic treatment could provide more precise information.

Now, performing a point transformation on the coordinate,

$$Q = q \exp\left(\frac{\gamma(t)}{2}\right) \tag{21}$$

a new Lagrange function is obtained,

$$\mathcal{L}(Q, \dot{Q}; t) = \frac{1}{2} m(t)\dot{Q}^2 + \frac{1}{2} m(t)\Xi^2(t) Q^2 - \frac{1}{2} m(t)\dot{\gamma}(t) Q\dot{Q} + QF(t)\exp\left(\frac{\gamma(t)}{2}\right) \tag{22}$$

where

$$\Xi^2(t) = \frac{\dot{\gamma}^2(t)}{4} - \omega_0^2. \tag{23}$$

The canonical momentum is

$$P = \frac{\partial\mathcal{L}}{\partial\dot{Q}} = m(t)\left(\dot{Q} - \frac{\dot{\gamma}(t)}{2} Q\right) = p\exp\left(\frac{\gamma(t)}{2}\right) \tag{24}$$

where $p_{\text{phys}} = m(t)\dot{q}$ is the physical momentum and q is the physical position, whereas P and Q are the mathematical canonical variables. As such the AD Hamiltonian is

$$H(P, Q; t) = \frac{P^2}{2m_0} \mathrm{e}^{-\zeta t} + \frac{1}{2} m_0\omega_0^2 Q^2 \mathrm{e}^{\zeta t} + \frac{\gamma_0}{2} PQ\,\mathrm{e}^{-\zeta t} - QF(t)\exp\left(\frac{\gamma(t)}{2}\right) \tag{25}$$

and as in the case of a pure dissipative Hamiltonian, it does not represent the energy; it is the generator of the motion of a TDM HO interacting with a heat reservoir.

The usual quantization of (25) gives

$$\boldsymbol{H}(t) = \frac{\boldsymbol{P}^2}{2m_0} \mathrm{e}^{-\zeta t} + \frac{1}{2} m_0\omega_0^2 \boldsymbol{Q}^2 \mathrm{e}^{\zeta t} + \frac{\gamma_0}{4}\{\boldsymbol{Q}, \boldsymbol{P}\}\,\mathrm{e}^{-\zeta t} - \boldsymbol{Q}F(t)\exp\left(\frac{\gamma(t)}{2}\right) \tag{26}$$

where the stochastic force continues as a c-number function. Finally, we recall that the physical quantities are evaluated, quantitatively, as double means (quantum and classical) over the products and sums of the physical operators,

$$\boldsymbol{q}_{\text{phys}} = \boldsymbol{Q}\exp\left(-\frac{\gamma(t)}{2}\right) \qquad \boldsymbol{p}_{\text{phys}} = \boldsymbol{P}\exp\left(-\frac{\gamma(t)}{2}\right) \tag{27}$$

which are stochastic.

4. The Heisenberg equations

From the Hamiltonian operator, (26), one has the following equations of motion for the canonical position and momentum operators

$$\frac{\mathrm{d}\boldsymbol{Q}_{\mathrm{H}}}{\mathrm{d}t} = \frac{\boldsymbol{P}_{\mathrm{H}}}{m_0}\,\mathrm{e}^{-\zeta t} + \frac{\gamma_0}{2}\boldsymbol{Q}_{\mathrm{H}}\,\mathrm{e}^{-\zeta t} \tag{28}$$

$$\frac{\mathrm{d}\boldsymbol{P}_{\mathrm{H}}}{\mathrm{d}t} = -m_0\omega_0^2\boldsymbol{Q}_{\mathrm{H}}\,\mathrm{e}^{\zeta t} - \frac{\gamma_0}{2}\boldsymbol{P}_{\mathrm{H}}\,\mathrm{e}^{-\zeta t} + F(t)\exp\left(\frac{\gamma(t)}{2}\right)\mathbf{1}. \tag{29}$$

Differentiating (28) with respect to t and using (29) one obtains the second-order differential equation,

$$\frac{\mathrm{d}^2\boldsymbol{Q}_{\mathrm{H}}}{\mathrm{d}t^2} + \zeta\frac{\mathrm{d}\boldsymbol{Q}_{\mathrm{H}}}{\mathrm{d}t} + \left(\omega_0^2 - \frac{\gamma_0^2}{4}\,\mathrm{e}^{-2\zeta t}\right)\boldsymbol{Q}_{\mathrm{H}} = \frac{F(t)}{m_0}\exp\left(\frac{\gamma(t)}{2} - \zeta t\right)\mathbf{1}. \tag{30}$$

Now, since (28) and (29) are linear, we assume the solution to be a sum of generators (in the Heisenberg picture) of the Weyl–Heisenberg algebra, multiplied by time-dependent coefficients,

$$\boldsymbol{Q}_{\mathrm{H}}(t) = u(t)\boldsymbol{Q}_0 + v(t)\boldsymbol{P}_0 + w(t)\mathbf{1}. \tag{31}$$

Substituting (31) into the q-number equation (30), this one splits into three uncoupled c-number second-order differential equations for the time-dependent coefficients,

$$\frac{\mathrm{d}^2u}{\mathrm{d}t^2} + \zeta\frac{\mathrm{d}u}{\mathrm{d}t} + \left(\omega_0^2 - \frac{\gamma_0^2}{4}\,\mathrm{e}^{-2\zeta t}\right)u = 0 \tag{32}$$

$$\frac{\mathrm{d}^2v}{\mathrm{d}t^2} + \zeta\frac{\mathrm{d}v}{\mathrm{d}t} + \left(\omega_0^2 - \frac{\gamma_0^2}{4}\,\mathrm{e}^{-2\zeta t}\right)v = 0 \tag{33}$$

$$\frac{\mathrm{d}^2w}{\mathrm{d}t^2} + \zeta\frac{\mathrm{d}w}{\mathrm{d}t} + \left(\omega_0^2 - \frac{\gamma_0^2}{4}\,\mathrm{e}^{-2\zeta t}\right)w = \frac{F(t)}{m_0}\exp\left(\frac{\gamma(t)}{2} - \zeta t\right) \tag{34}$$

with the following initial conditions,

$$u(0) = 1 \qquad \left.\frac{\mathrm{d}u}{\mathrm{d}t}\right|_{t=0} = \frac{\gamma_0}{2} \tag{35}$$

$$v(0) = 0 \qquad \left.\frac{\mathrm{d}v}{\mathrm{d}t}\right|_{t=0} = \frac{1}{m_0} \tag{36}$$

$$w(0) = 0 \qquad \left.\frac{\mathrm{d}w}{\mathrm{d}t}\right|_{t=0} = 0. \tag{37}$$

According to the appendix, the solutions to (32)–(34) are

$$\begin{aligned} u(t) = \mathrm{e}^{-\zeta t/2}\Bigg\{&\left[\left(\nu - \frac{\gamma_0}{2\zeta} - \frac{1}{2}\right)\mathcal{K}_\nu\left(\frac{\gamma_0}{2\zeta}\right) + \frac{\gamma_0}{2\zeta}\mathcal{K}_{\nu-1}\left(\frac{\gamma_0}{2\zeta}\right)\right]\mathcal{I}_\nu\left(\frac{\gamma_0}{2\zeta}\,\mathrm{e}^{-\zeta t}\right) \\ &-\left[\left(\nu - \frac{\gamma_0}{2\zeta} - \frac{1}{2}\right)\mathcal{I}_\nu\left(\frac{\gamma_0}{2\zeta}\right) - \frac{\gamma_0}{2\zeta}\mathcal{I}_{\nu-1}\left(\frac{\gamma_0}{2\zeta}\right)\right]\mathcal{K}_\nu\left(\frac{\gamma_0}{2\zeta}\,\mathrm{e}^{-\zeta t}\right)\Bigg\} \end{aligned} \tag{38}$$

$$v(t) = \frac{1}{m_0\zeta}\,\mathrm{e}^{-\zeta t/2}\left[\mathcal{I}_\nu\left(\frac{\gamma_0}{2\zeta}\right)\mathcal{K}_\nu\left(\frac{\gamma_0}{2\zeta}\,\mathrm{e}^{-\zeta t}\right) - \mathcal{K}_\nu\left(\frac{\gamma_0}{2\zeta}\right)\mathcal{I}_\nu\left(\frac{\gamma_0}{2\zeta}\,\mathrm{e}^{-\zeta t}\right)\right] \tag{39}$$

$$\begin{aligned} w(t) = \frac{1}{m_0\zeta}\,\mathrm{e}^{-\zeta t/2}&\int_0^t \exp\left(\frac{1}{2}[\gamma(t_1) - \zeta t_1]\right)F(t_1) \\ &\times\left[\mathcal{I}_\nu\left(\frac{\gamma_0}{2\zeta}\,\mathrm{e}^{-\zeta t}\right)\mathcal{K}_\nu\left(\frac{\gamma_0}{2\zeta}\,\mathrm{e}^{-\zeta t_1}\right) - \mathcal{K}_\nu\left(\frac{\gamma_0}{2\zeta}\,\mathrm{e}^{-\zeta t}\right)\mathcal{I}_\nu\left(\frac{\gamma_0}{2\zeta}\,\mathrm{e}^{-\zeta t_1}\right)\right]\mathrm{d}t_1 \end{aligned} \tag{40}$$

where $\mathcal{I}_\nu(x)$ and $\mathcal{K}_\nu(x)$ are the modified Bessel functions of first and third kind, respectively. The dimensionless order is $\nu = \tilde{\Omega}_0/\zeta$, where $\tilde{\Omega}_0^2 = \zeta^2/4 - \omega_0^2$.

Now, isolating the momentum operator in (28) and using (31), one verifies that similarly to the position operator in the Heisenberg picture, the momentum operator can be written as

$$\boldsymbol{P}_{\mathrm{H}}(t) = \mu(t)\boldsymbol{Q}_0 + \kappa(t)\boldsymbol{P}_0 + \eta(t)\mathbf{1} \tag{41}$$

where the new time-dependent coefficients are related to the old ones by the following equations,

$$\mu(t) \equiv m_0 \left[\mathrm{e}^{\zeta t} \frac{\mathrm{d}u(t)}{\mathrm{d}t} - \frac{\gamma_0}{2} u(t) \right] \tag{42}$$

$$\kappa(t) \equiv m_0 \left[\mathrm{e}^{\zeta t} \frac{\mathrm{d}v(t)}{\mathrm{d}t} - \frac{\gamma_0}{2} v(t) \right] \tag{43}$$

$$\eta(t) \equiv m_0 \left[\mathrm{e}^{\zeta t} \frac{\mathrm{d}w(t)}{\mathrm{d}t} - \frac{\gamma_0}{2} w(t) \right]. \tag{44}$$

The commutation relations $[\boldsymbol{Q}_{\mathrm{H},i}(t), \boldsymbol{P}_{\mathrm{H},j}(t)] = \mathrm{i}\hbar\delta_{i,j}\mathbf{1}$ will hold only for

$$u(t)\kappa(t) - v(t)\mu(t) = 1. \tag{45}$$

This condition can be easily verified. Multiplying (42) by $v(t)$ and (43) by $u(t)$ and subtracting the first from the second

$$u(t)\kappa(t) - v(t)\mu(t) = m_0\, \mathrm{e}^{\zeta t} \Lambda(t) \tag{46}$$

where

$$\Lambda(t) \equiv u(t)\dot{v}(t) - v(t)\dot{u}(t). \tag{47}$$

But from (32) and (33) we have the first-order differential equation

$$\frac{\mathrm{d}\Lambda(t)}{\mathrm{d}t} + \zeta\,\Lambda(t) = 0 \tag{48}$$

the solution of which is

$$\Lambda(t) = \Lambda(0)\,\mathrm{e}^{-\zeta t} = \frac{1}{m_0}\,\mathrm{e}^{-\zeta t} \tag{49}$$

where the second equality follows from (35) and (36). Now, substituting (49) into (46) we obtain (45). Therefore, the initial conditions have a fundamental importance to ensure the commutation relation and consequently, to preserve the Heisenberg uncertainty relation.

5. Mean energy and position–momentum uncertainty relations for $0 < \zeta/2 \leqslant \omega_0$

5.1. The quantized CK Hamiltonian

For the sake of comparision we shall first consider the pure CK amplifying Hamiltonian. Since the derivation is quite simple and it has already been performed, see [9] and references therein, we will only write down the expressions for the mean energy and variances of momentum and position.

Assuming the under-amplified situation, $0 < \zeta/2 < \omega_0$, one verifies that the mean energy, in a coherent state with $\alpha_{\mathrm{R}} = \alpha_{\mathrm{I}}$, oscillates in time,

$$\langle \boldsymbol{E}_{\mathrm{CK}} \rangle = \left(1 + \frac{\zeta}{\omega_0 - \zeta/2} \sin^2 \Omega_0 t \right) E_0 - \frac{\zeta\,\omega_0}{2\Omega_0^2} \sin^2 \Omega_0 t \tag{50}$$

and the average over one period of oscillation gives

$$\overline{\langle \boldsymbol{E}_{\mathrm{CK}} \rangle}^{T} = \frac{\omega_0}{\omega_0 - \zeta/2} \left(E_0 - \frac{1}{2} \frac{\zeta/2}{\omega_0 + \zeta/2} \right). \tag{51}$$

Thus, the average energy is larger than its initial value E_0, and the term in parenthesis is never negative even if the initial state is the ground state, $E_0 = 1/2$. However, at critical amplification, $\omega_0 = \zeta/2$, the behaviour is quite different; the CK Hamiltonian shows that as time goes on the mean energy increases at a rate proportional to t^2,

$$\langle \boldsymbol{E}_{\mathrm{CK}} \rangle_{\mathrm{c}} = E_0 + (E_0 - \tfrac{1}{4})\zeta^2 t^2. \tag{52}$$

The variances for arbitrary operators $\boldsymbol{A}$ and $\boldsymbol{B}$ are defined as $\sigma_A \equiv [\langle \boldsymbol{A}^2 \rangle - \langle \boldsymbol{A} \rangle^2]$ and $\sigma_{AB} \equiv \left[\langle \frac{1}{2} \{\boldsymbol{A}, \boldsymbol{B}\} \rangle - \langle \boldsymbol{A} \rangle \langle \boldsymbol{B} \rangle \right]$, and the Robertson-Schrödinger (RS) [28] relation is

$$\Delta_{AB} \equiv \sigma_A \sigma_{\mathrm{B}} - (\sigma_{AB})^2 \geqslant \frac{\hbar^2}{4}. \tag{53}$$

Then, for the position and momentum operators, and for $\omega_0 \neq \zeta/2$, one has

$$\sigma_q^{\mathrm{CK}} = \frac{\hbar}{2m_0\omega_0} \mathrm{e}^{-\zeta t} \left[1 + \frac{\zeta^2}{2\Omega_0^2} \sin^2 \Omega_0 t + \frac{\zeta}{2\Omega_0} \sin 2\Omega_0 t \right] \tag{54}$$

$$\sigma_p^{\mathrm{CK}} = \frac{\hbar m_0 \omega_0}{2} \mathrm{e}^{\zeta t} \left[1 + \frac{\zeta^2}{2\Omega_0^2} \sin^2 \Omega_0 t - \frac{\zeta}{2\Omega_0} \sin 2\Omega_0 t \right] \tag{55}$$

$$\sigma_{qp}^{\mathrm{CK}} = -\frac{\hbar \omega_0 \zeta}{2\Omega_0^2} \sin^2 \Omega_0 t \tag{56}$$

where the squeezing and stretching of variances σ_q^{CK} and σ_p^{CK} occur owing to the TD exponential factors, and the behaviour of variances interchanges by changing the sign of ζ, $\zeta \to -\zeta$. The crossed-variance $\sigma_{qp}^{\mathrm{CK}}$ displays an oscillating correlation between the variables, therefore mass increase in an HO results in a correlation of the variables. Now, considering the product of variances,

$$\sigma_q^{\mathrm{CK}} \sigma_p^{\mathrm{CK}} = \frac{\hbar^2}{4} \left[1 + \frac{\zeta^2 \omega_0^2}{\Omega_0^4} \sin^4 \Omega_0 t \right] \geqslant \frac{\hbar^2}{4} \tag{57}$$

$$(\sigma_{qp}^{\mathrm{CK}})^2 = \frac{\hbar^2}{4} \frac{\zeta^2 \omega_0^2}{\Omega_0^4} \sin^4 \Omega_0 t \tag{58}$$

we verify that the Heisenberg inequality is satisfied and the RS expression is an invariant [29],

$$\Delta_{qp}^{\mathrm{CK}} = \hbar^2/4. \tag{59}$$

When strong oscillations occur and the equipment responds more slowly, only average values are recorded, thus the averaging of (57) and (58) over one period of oscillation, for $\Omega_0 \neq 0$, gives

$$\overline{\sigma_q^{\mathrm{CK}} \sigma_p^{\mathrm{CK}}}^{T} = \frac{\hbar^2}{4} \left(1 + \frac{3\zeta^2 \omega_0^2}{8\Omega_0^4} \right) \geqslant \frac{\hbar^2}{4} \tag{60}$$

$$\overline{(\sigma_{qp}^{\mathrm{CK}})^2}^{T} = \frac{\hbar^2}{4} \frac{3\zeta^2 \omega_0^2}{8\Omega_0^4}. \tag{61}$$

Moreover, it is easier to compare the above expressions to the variances, to be presented in section 5.2.

At critical amplification ($\Omega_0 = 0$) the variances becomes

$$\sigma_{q,\mathrm{c}}^{\mathrm{CK}} = \frac{\hbar}{2m_0\omega_0}\,\mathrm{e}^{-\zeta t}\left(1 + \zeta t + \frac{1}{2}\zeta^2 t^2\right) \tag{62}$$

$$\sigma_{p,\mathrm{c}}^{\mathrm{CK}} = \frac{\hbar m_0\omega_0}{2}\,\mathrm{e}^{\zeta t}\left(1 - \zeta t + \frac{1}{2}\zeta^2 t^2\right) \tag{63}$$

$$\sigma_{qp,\mathrm{c}}^{\mathrm{CK}} = -\frac{\hbar}{4}\zeta^2 t^2 \tag{64}$$

where the polynomials in parentheses do not change the asymptotic behaviour of the squeezing or stretching of variances. However, as the system evolves, their product increases as t^4,

$$\sigma_{q,\mathrm{c}}^{\mathrm{CK}}\sigma_{p,\mathrm{c}}^{\mathrm{CK}} = \frac{\hbar^2}{4}\left(1 + \frac{1}{4}\zeta^4 t^4\right) \tag{65}$$

$$\left(\sigma_{qp,\mathrm{c}}^{\mathrm{CK}}\right)^2 = \frac{\hbar^2}{4}\frac{\zeta^4 t^4}{4} \tag{66}$$

independently of the sign of ζ, but the RS relation remains constant, (59).

5.2. *The AD Hamiltonian, solutions for weak damping* ($\gamma_0/\zeta \ll 1$)

From the general solutions of section 4, we concentrate on the analysis of the mean values of physical quantities such as the physical position and momentum, their variances and energy for the regimes of under and critical amplification $0 < \zeta/2 \leqslant \omega_0$, but for the *weak damping* limit $\gamma_0/\zeta \ll 1$. In this limit the parameter $\tilde{\Omega}_0$ and the order ν become purely imaginary; however, for positive arguments and real positive index, the modified Bessel functions behave as [30],

$$\mathcal{I}_\nu\left(\frac{\gamma_0}{2\zeta}\,\mathrm{e}^{-\zeta t}\right) \approx \frac{1}{2^\nu\Gamma(\nu+1)}\left(\frac{\gamma_0}{2\zeta}\right)^\nu \mathrm{e}^{-\tilde{\Omega}_0 t} \tag{67}$$

$$\mathcal{K}_\nu\left(\frac{\gamma_0}{2\zeta}\,\mathrm{e}^{-\zeta t}\right) \approx 2^{\nu-1}\Gamma(\nu)\left(\frac{2\zeta}{\gamma_0}\right)^\nu \mathrm{e}^{\tilde{\Omega}_0 t}. \tag{68}$$

Since in our case ν is purely imaginary, by analytic continuation the above approximations can be used, allowing us to obtain, up to the order γ_0/ζ, the position and momentum operators,

$$\begin{aligned}\boldsymbol{Q}_{\mathrm{H}}(t) = {}& \mathrm{e}^{-\zeta t/2}\Bigg[\left(\cos\Omega_0 t + \frac{\gamma_0+\zeta}{2\Omega_0}\sin\Omega_0 t\right)\boldsymbol{Q}_0 + \left(\frac{1}{m_0\Omega_0}\sin\Omega_0 t\right)\boldsymbol{P}_0 \\ & - \frac{1}{m_0\Omega_0}\,\mathrm{e}^{\gamma(t)/2}\int_0^t \exp\left(\frac{-\zeta t_1}{2}\right)\sin[\Omega_0(t-t_1)]F(t_1)\,\mathrm{d}t_1\mathbf{1}\Bigg]\end{aligned} \tag{69}$$

$$\begin{aligned}\boldsymbol{P}_{\mathrm{H}}(t) = {}& \mathrm{e}^{\zeta t/2}\Bigg\{\left(-\frac{m_0\omega_0^2}{\Omega_0}\sin\Omega_0 t\right)\boldsymbol{Q}_0 + \left(\cos\Omega_0 t - \frac{\gamma_0+\zeta}{2\Omega_0}\sin\Omega_0 t\right)\boldsymbol{P}_0 \\ & + \mathrm{e}^{\gamma(t)/2}\int_0^t \exp\left(-\frac{\zeta t_1}{2}\right)\left[\frac{\zeta}{2\Omega_0}\sin[\Omega_0(t-t_1)] - \cos[\Omega_0(t-t_1)]\right]F(t_1)\,\mathrm{d}t_1\mathbf{1}\Bigg\}\end{aligned} \tag{70}$$

where $\Omega_0^2 = \omega_0^2 - \frac{1}{4}\zeta^2 = -\tilde{\Omega}_0^2$.

The energy operator defined as

$$\boldsymbol{E}_{\mathrm{AD}}(t) \equiv \frac{\boldsymbol{p}_{\mathrm{phys}}^2}{2m(t)} + \frac{1}{2}m(t)\omega_0^2\boldsymbol{q}_{\mathrm{phys}}^2 = \exp[-\gamma(t)]\left[\frac{\boldsymbol{P}_{\mathrm{H}}^2}{2m(t)} + \frac{1}{2}m(t)\omega_0^2\boldsymbol{Q}_{\mathrm{H}}^2\right] \tag{71}$$

is a stochastic operator, so, in order to obtain the mean energy, we first substitute (69) and (70) into (71), calculate the mean value in a coherent-state $|\alpha\rangle$, with $\alpha_R = \alpha_I$, and then take the average over the stochastic force,

$$\langle\langle \boldsymbol{E}_{\rm AD}(t)\rangle\rangle = \left\{\left[(u(t)+m_0\omega_0 v(t))^2\,{\rm e}^{\zeta t} + \left(\kappa(t)+\frac{\mu(t)}{m_0\omega_0}\right)^2{\rm e}^{-\zeta t}\right]\frac{E_0}{2} - \frac{1}{2}\left[m_0\omega_0 u(t)v(t)\,{\rm e}^{\zeta t} + \frac{\mu(t)\kappa(t)}{m_0\omega_0}\,{\rm e}^{-\zeta t}\right] + \frac{1}{2}\left[m_0\omega_0\langle w^2(t)\rangle\,{\rm e}^{\zeta t} + \frac{\langle\eta^2(t)\rangle}{m_0\omega_0}\,{\rm e}^{-\zeta t}\right]\right\}{\rm e}^{-\gamma(t)} \tag{72}$$

thus resulting in the following time-oscillating mean energy,

$$\langle\langle \boldsymbol{E}_{\rm AD}(t)\rangle\rangle = \left[\left(1+\frac{\zeta}{\omega_0-\zeta/2}f_0\sin^2\Omega_0 t\right)E_0 - \frac{\zeta\omega_0}{2\Omega_0^2}f_0\sin^2\Omega_0 t\right]{\rm e}^{-\gamma(t)} + \frac{\Omega_0^2}{\omega_0^2}\frac{\gamma_0}{2\zeta}\coth\left(\frac{\hbar\omega_0}{2k_{\rm B}T}\right)\left[(1-{\rm e}^{-\zeta t}) + \frac{\zeta^2}{2\Omega_0^2}\sin^2\Omega_0 t\right] \tag{73}$$

where the double brackets stands for the ensemble average and quantum mean. The dimensionless energy is written in units of $\hbar\omega_0$, E_0 is the initial energy and $f_0 = 1+\gamma_0/\zeta$. Asymptotically the system attains a stationary state with mean energy,

$$\lim_{t\to\infty}\langle\langle \boldsymbol{E}_{\rm AD}(t)\rangle\rangle = \left[\left(1+\frac{\zeta}{\omega_0-\zeta/2}f_0\sin^2\Omega_0 t\right)E_0 - \frac{\zeta\omega_0}{2\Omega_0^2}f_0\sin^2\Omega_0 t\right]{\rm e}^{-\gamma_0/\zeta} + \frac{\Omega_0^2}{\omega_0^2}\frac{\gamma_0}{2\zeta}\coth\left(\frac{\hbar\omega_0}{2k_{\rm B}T}\right)\left(1+\frac{\zeta^2}{2\Omega_0^2}\sin^2\Omega_0 t\right). \tag{74}$$

For $\omega_0 \neq \zeta/2$, taking the average over one period of oscillation in (74), and keeping only terms linear in γ_0/ζ, we get

$$\overline{\langle\langle \boldsymbol{E}_{\rm AD}\rangle\rangle}^T = \overline{\langle \boldsymbol{E}_{\rm CK}\rangle}^T - \frac{\gamma_0}{\zeta}\left[E_0 - \frac{1}{2}\coth\left(\frac{\hbar\omega_0}{2k_{\rm B}T}\right)\right] \tag{75}$$

where $\overline{\langle \boldsymbol{E}_{\rm CK}\rangle}^T$ is given by (51). As expected, if the initial energy is higher than the thermalization energy, $E_0 > \frac{1}{2}\coth(\hbar\omega_0/2k_{\rm B}T)$, then owing to dissipation, the mean asymptotic energy of the TDM HO stabilizes at a value that is lower than that for the CK Hamiltonian (51). Otherwise, the oscillator gains energy not only as a result of the mass increase but also by receiving energy from the thermalized environment, $E_0 < \frac{1}{2}\coth(\hbar\omega_0/2k_{\rm B}T)$. It is worth paying attention to the fact that the mean energy (75) depends on both parameters of the reservoir, the specific damping constant γ_0 and the temperature T, whereas in the case of pure damping (constant mass) the mean asymptotic energy depends only on the temperature (section 5.3). At the critical value, $\Omega_0 = 0$, the TDM HO has its energy increased at a rate proportional to t^2, thus, for $\gamma_0/\zeta \ll 1$, asymptotically, it will not feel the environment and the energy will coincide with (52).

The variances of the physical position and momentum can be written as

$$\sigma_q^{\rm AD} = \frac{\hbar}{2m_0\omega_0}\exp[-\gamma(t)][u^2(t) + m_0^2\omega_0^2 v^2(t) + 2m_0\omega_0\langle w^2(t)\rangle] \tag{76}$$

$$\sigma_p^{\rm AD} = \frac{\hbar m_0\omega_0}{2}\exp[-\gamma(t)]\left[\kappa^2(t) + \frac{\mu^2(t)}{m_0^2\omega_0^2} + \frac{2}{m_0\omega_0}\langle\eta^2(t)\rangle\right] \tag{77}$$

$$\sigma_{qp}^{\rm AD} = \exp[-\gamma(t)]\left[\frac{\hbar}{2m_0\omega_0}u(t)\mu(t) + \frac{m_0\omega_0}{2}v(t)\kappa(t) + \langle w(t)\eta(t)\rangle\right] \tag{78}$$

and for $\gamma_0 \ll \zeta$ we have

$$\sigma_q^{\rm AD} = \left(\frac{\hbar}{2m_0\omega_0}\right) {\rm e}^{-[\gamma(t)+\zeta t]} \left\{1 + \left[\frac{\zeta^2}{2\Omega_0^2}\sin^2\Omega_0 t + \frac{\zeta}{2\Omega_0}\sin 2\Omega_0 t\right] f_0 + \frac{\gamma_0}{\zeta}\,{\rm e}^{\gamma(t)} \times\left[(1-{\rm e}^{-\zeta t}) + \frac{\zeta^2}{2\Omega_0^2}\sin^2\Omega_0 t - \frac{\zeta}{2\Omega_0}\sin 2\Omega_0 t\right]\coth\left(\frac{\hbar\omega_0}{2k_{\rm B}T}\right)\right\} \tag{79}$$

$$\sigma_p^{\rm AD} = \left(\frac{\hbar m_0\omega_0}{2}\right) {\rm e}^{-[\gamma(t)-\zeta t]} \left\{1 + \left[\frac{\zeta^2}{2\Omega_0^2}\sin^2\Omega_0 t - \frac{\zeta}{2\Omega_0}\sin 2\Omega_0 t\right] f_0 + \frac{\gamma_0}{\zeta}\,{\rm e}^{\gamma(t)} \times\left[(1-{\rm e}^{-\zeta t}) + \frac{\zeta^2}{2\Omega_0^2}\sin^2\Omega_0 t + \frac{\zeta}{2\Omega_0}\sin 2\Omega_0 t\right]\coth\left(\frac{\hbar\omega_0}{2k_{\rm B}T}\right)\right\} \tag{80}$$

$$\sigma_{qp}^{\rm AD} = -\frac{\hbar}{2}\left\{\frac{\omega_0\zeta f_0}{\Omega_0^2}\,{\rm e}^{-\gamma(t)}\sin^2\Omega_0 t + \frac{\gamma_0\Omega_0^2}{\omega_0^3}\left[(1-{\rm e}^{-\zeta t}) - \left(1-\frac{\zeta^2}{4\Omega_0^2}\right)\sin^2\Omega_0 t - \frac{\zeta}{2\Omega_0}\sin 2\Omega_0 t\right]\coth\left(\frac{\hbar\omega_0}{2k_{\rm B}T}\right)\right\}. \tag{81}$$

Comparing (54) and (55) with (79) and (80), one verifies that squeezing occurs at the same rate, although here there is an additional (lowest order-correction) term in the braces, owing to the environment.

Now, taking the asymptotic value of the product $\sigma_q^{\rm AD}\sigma_p^{\rm AD}$ and $(\sigma_{qp}^{\rm AD})^2$, which does not depend on the TD exponential factors, and then considering terms up to first order in γ_0/ζ, one finds

$$\sigma_q^{\rm AD}\sigma_p^{\rm AD} = \frac{\hbar^2}{4}\left\{1 + \frac{\omega_0^2\zeta^2}{\Omega_0^4}\sin^4\Omega_0 t - \frac{2\gamma_0}{\zeta}\left(1+\frac{\zeta^2}{2\Omega_0^2}\sin^2\Omega_0 t\right) + \frac{2\gamma_0}{\zeta}\left[1 + \frac{2\zeta^2}{\Omega_0^2}\sin^2\Omega_0 t - \frac{\zeta^2}{\Omega_0^2}\left(1-\frac{\zeta^2}{4\Omega_0^2}\right)\sin^4\Omega_0 t\right] \times\coth\left(\frac{\hbar\omega_0}{2k_{\rm B}T}\right)\right\} \geqslant \frac{\hbar^2}{4} \tag{82}$$

$$(\sigma_{qp}^{\rm AD})^2 = \frac{\hbar^2}{4}\left\{\frac{\zeta^2\omega_0^2}{\Omega_0^4}\sin^4\Omega_0 t + \frac{2\gamma_0}{\zeta}\frac{\zeta^2}{\omega_0^2}\sin^2\Omega_0 t \times\left[1 - \frac{\zeta}{2\Omega_0}\sin 2\Omega_0 t - \left(1-\frac{\zeta^2}{4\Omega_0^2}\right)\sin^2\Omega_0 t\right]\coth\left(\frac{\hbar\omega_0}{2k_{\rm B}T}\right)\right\}. \tag{83}$$

Consequently, the RS expression is

$$\Delta^{\rm AD} = \frac{\hbar^2}{4}\left\{1 - \frac{2\gamma_0}{\zeta}\left(1+\frac{\zeta^2}{2\Omega_0^2}\sin^2\Omega_0 t\right) + \frac{2\gamma_0}{\zeta}\left[1 + \frac{\zeta^2}{\Omega_0^2}\left(1+\frac{\zeta^2}{4\omega_0^2}\right)\sin^2\Omega_0 t + \frac{\zeta^3}{2\Omega_0\omega_0^2}\sin^2\Omega_0 t\sin 2\Omega_0 t - \frac{\zeta^4}{4\Omega_0^4}\left(1-\frac{\zeta^2}{2\omega_0^2}\right)\sin^4\Omega_0 t\right] \times\coth\left(\frac{\hbar\omega_0}{2k_{\rm B}T}\right)\right\} \geqslant \frac{\hbar^2}{4}. \tag{84}$$

However, although the Heisenberg and RS inequalities are satisfied, as in the preceding subsection, they are easier to view when considering the average over one period of oscillation for $\Omega_0 \neq 0$,

$$\overline{\sigma_q^{\rm AD}\sigma_p^{\rm AD}}^T = \frac{\hbar^2}{4}\left\{1 + \frac{3\omega_0^2\zeta^2}{8\Omega_0^4} - \frac{2\gamma_0}{\zeta}\left[\frac{\omega_0^2}{\Omega_0^2} - \left(1+\frac{5\zeta^2}{8\Omega_0^2}+\frac{3\zeta^4}{32\Omega_0^4}\right)\coth\left(\frac{\hbar\omega_0}{2k_{\rm B}T}\right)\right]\right\} \tag{85}$$

$$\overline{(\sigma_{qp}^{\mathrm{AD}})^2}^T = \frac{\hbar^2}{4}\left\{\frac{3\zeta^2\omega_0^2}{8\Omega_0^4} + \frac{2\gamma_0}{\zeta}\frac{\zeta^2}{\Omega_0^2}\left(1+\frac{\zeta^2}{2\omega_0^2}\right)\coth\left(\frac{\hbar\omega_0}{2k_{\mathrm{B}}T}\right)\right\} \tag{86}$$

$$\overline{\Delta^{\mathrm{AD}}}^T = \frac{\hbar^2}{4}\left\{1 + \frac{2\gamma_0}{\zeta}\left[\left(1+\frac{\zeta^2}{2\Omega_0^2}+\frac{\zeta^4}{32\Omega_0^4}+\frac{\zeta^6}{64\Omega_0^4\omega_0^2}\right)\coth\left(\frac{\hbar\omega_0}{2k_{\mathrm{B}}T}\right) - \frac{\omega_0^2}{\Omega_0^2}\right]\right\}. \tag{87}$$

Even when the TDM HO interacts with the vacuum, at $T = 0$ K, the parameter γ_0 is still present in the products of variances, contributing with an additional term in (85), in comparison with (60),

$$\overline{\sigma_q^{\mathrm{AD}}\sigma_p^{\mathrm{AD}}}^T(0\ \mathrm{K}) = \frac{\hbar^2}{4}\left[1+\frac{3\omega_0^2\zeta^2}{8\Omega_0^4}+\frac{2\gamma_0}{\zeta}\frac{3\zeta^2\omega_0^2}{8\Omega_0^4}\right] \geqslant \frac{\hbar^2}{4} \tag{88}$$

$$\overline{\Delta^{\mathrm{AD}}}^T(0\ \mathrm{K}) = \frac{\hbar^2}{4}\left[1+\frac{2\gamma_0}{\zeta}\frac{\zeta^2}{4\Omega_0^2}\left(1+\frac{\zeta^2}{8\Omega_0^2}+\frac{\zeta^4}{16\omega_0^2\Omega_0^2}\right)\right] \geqslant \frac{\hbar^2}{4}. \tag{89}$$

5.3. *Recovering the case of the dissipative HO*

Considering the purely dissipative HO, $\zeta = 0$ in (30), one finds

$$\boldsymbol{Q}_{\mathrm{H}}(t) = \left(\cos\Omega t + \frac{\gamma_0}{2\Omega}\sin\Omega t\right)\boldsymbol{Q}_0 + \left(\frac{1}{m_0\Omega}\sin\Omega t\right)\boldsymbol{P}_0 - \frac{1}{m_0\Omega}\int_0^t \exp\left(\frac{\gamma_0 t_1}{2}\right)\sin[\Omega(t-t_1)]F(t_1)\,\mathrm{d}t_1\mathbf{1} \tag{90}$$

$$\boldsymbol{P}_{\mathrm{H}}(t) = \left(-\frac{m_0\omega_0^2}{\Omega}\sin\Omega t\right)\boldsymbol{Q}_0 + \left(\cos\Omega t - \frac{\gamma_0}{2\Omega}\sin\Omega t\right)\boldsymbol{P}_0 + \int_0^t \exp\left(\frac{\gamma_0 t_1}{2}\right)\left[\frac{\gamma_0}{2\Omega}\sin[\Omega(t-t_1)] - \cos[\Omega(t-t_1)]\right]F(t_1)\,\mathrm{d}t_1\mathbf{1} \tag{91}$$

where $\Omega^2 = \omega_0^2 - \gamma_0^2/4$. This case is identical to the original Svin'in treatment of a linear HO with friction [9, 18].

Following the same steps as in section 5.2, taking the quantum mean energy in a coherent state and the ensemble average, we obtain

$$\langle\langle \boldsymbol{E}(t)\rangle\rangle = E_0\exp(-\gamma_0 t) + \frac{1}{2}\coth\left(\frac{\hbar\omega_0}{2k_{\mathrm{B}}T}\right)[1-\exp(-\gamma_0 t)] \tag{92}$$

as expected, and the thermalized asymptotic value is

$$\lim_{t\to\infty}\langle\langle \boldsymbol{E}(t)\rangle\rangle = \frac{1}{2}\coth\left(\frac{\hbar\omega_0}{2k_{\mathrm{B}}T}\right) \tag{93}$$

which does not depend on the parameter γ_0, as in the case of the asymptotic mean energy of the TDM HO (75).

Now, using calculations similar to those of the previous subsection, one obtains the variances

$$\sigma_q = \frac{\hbar}{2m_0\omega_0}\,\mathrm{e}^{-\gamma_0 t}\left\{1+\frac{\gamma_0^2}{2\Omega^2}\sin^2\Omega t + \frac{\gamma_0}{2\Omega}\sin 2\Omega t + \left[(\mathrm{e}^{\gamma_0 t}-1) - \frac{\gamma_0^2}{2\Omega^2}\sin^2\Omega t - \frac{\gamma_0}{2\Omega}\sin 2\Omega t\right]\coth\left(\frac{\hbar\omega_0}{2k_{\mathrm{B}}T}\right)\right\} \tag{94}$$

$$\sigma_p = \frac{\hbar m_0\omega_0}{2}\,\mathrm{e}^{-\gamma_0 t}\left\{1 + \frac{\gamma_0^2}{2\Omega^2}\sin^2\Omega t - \frac{\gamma_0}{2\Omega}\sin 2\Omega t + \left[(\mathrm{e}^{\gamma_0 t} - 1) - \frac{\gamma_0^2}{2\Omega^2}\sin^2\Omega t + \frac{\gamma_0}{2\Omega}\sin 2\Omega t\right]\coth\left(\frac{\hbar\omega_0}{2k_B T}\right)\right\} \tag{95}$$

$$\sigma_{qp} = \frac{\hbar\omega_0\gamma_0}{2\Omega^2}\left[\coth\left(\frac{\hbar\omega_0}{2k_B T}\right) - 1\right]\mathrm{e}^{-\gamma_0 t}\sin^2\Omega t \tag{96}$$

and their asymptotic values are those of a thermalized HO,

$$\lim_{t\to\infty}\sigma_q = \frac{\hbar}{2m_0\omega_0}\coth\left(\frac{\hbar\omega_0}{2k_B T}\right) \tag{97}$$

$$\lim_{t\to\infty}\sigma_p = \frac{\hbar m_0\omega_0}{2}\coth\left(\frac{\hbar\omega_0}{2k_B T}\right) \tag{98}$$

$$\lim_{t\to\infty}\sigma_{qp} = 0 \tag{99}$$

that also do not depend on γ_0. Since thermalization destroys the p–q correlation (99), the Heisenberg product of variances goes to (5.25) of [9],

$$\lim_{t\to\infty}\sigma_q\sigma_p = \frac{\hbar^2}{4}\left[\coth\left(\frac{\hbar\omega_0}{2k_B T}\right)\right]^2. \tag{100}$$

6. Summary and conclusions

We recall that, classically, the CK Hamiltonian can correctly describe either a dissipative HO or a TDM HO; the distinction is made in the definition of the physical variables: position, linear momentum and energy. The quantization of that Hamiltonian and the subsequent solution of the Heisenberg equation of motion is compatible with the principles of quantum mechanics only when the CK Hamiltonian describes a TDM HO. For the quantum dissipative HO, a phenomenological TD stochastic force must be introduced in the equations of motion (in order to take into account the effects of quantum fluctuations produced by the environment) and, as prescribed by Havas, an integrating factor must be introduced in the Lagrange function, before performing the quantization. Care must be taken in distinguishing between the mathematical operators and the physical ones, since they do not coincide.

Here we considered an HO having its energy amplified by a TDM, and also being subject to a friction force due to the interaction with the environment. We first verified that the exponential integrating factor of the Lagrange function has an argument that is no longer linear in time, as in the case of a purely dissipative HO Hamiltonian. Then, obtaining the Hamilton function we quantized it and solved exactly the Heisenberg equation of motion for canonical position and momentum operators. Next we used these stochastic operators to calculate the mean energy and the variances of momentum and position in the under-amplified regime and weak-damping limit. The asymptotic expressions were displayed verifying the squeezing of variances together with a time-oscillation. We also verified that the Heisenberg and Robertson–Schrödinger inequalities are satisfied. Finally, it is worth noting that the thermalized energy, (93), and the product of variances, (100), depend essentially on the temperature of the reservoir, whereas the stationary values of the TDM HO depend (besides, obviously, on the amplifying parameter ζ) on the friction constant γ_0. Therefore, in order to determine the parameter of the environment γ_0, out from the asymptotic values of the mean energy or from the variances, its becomes necessary to consider a mass-accreting HO. In conclusion we showed that it is possible to construct a Hamiltonian that gives a quantal description of a dissipative mass-accreting HO.

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Appendix. Solutions of the Heisenberg equations

We shall look for the analytic solution of the differential equation (32),

$$\frac{d^2u}{dt^2} + \zeta\frac{du}{dt} + \left(\omega_0^2 - \frac{\gamma_0^2}{4}\,e^{-2\zeta t}\right)u = 0. \tag{101}$$

In order to simplify (101), we first introduce the new function $\tilde{u}(t)$

$$u(t) = e^{-\zeta t/2}\tilde{u}(t) \tag{102}$$

and so obtain

$$\frac{d^2\tilde{u}}{dt^2} - \left(\tilde{\Omega}_0^2 + \frac{\gamma_0^2}{4}\,e^{-2\zeta t}\right)\tilde{u} = 0 \tag{103}$$

where $\tilde{\Omega}_0^2 = \frac{1}{4}\zeta^2 - \omega_0^2$. Second, we do a change of variables, by introducing the new dimensionless variable

$$x = \frac{\gamma_0}{2\zeta}\,e^{-\zeta t}$$

and rewrite (103) as

$$\frac{d^2\tilde{u}}{dx^2} + \frac{1}{x}\frac{d\tilde{u}}{dx} - \left(1 + \frac{\nu^2}{x^2}\right)\tilde{u} = 0 \tag{104}$$

with $\nu^2 = \tilde{\Omega}_0{}^2/\zeta^2$. The solutions of (104) are found in the literature [30], namely,

$$\tilde{u}(x) = A_1\mathcal{I}_\nu(x) + A_2\mathcal{K}_\nu(x) \tag{105}$$

where $\mathcal{I}_\nu(x)$ and $\mathcal{K}_\nu(x)$ are the modified Bessel functions of the first and third kind, respectively. The dimensionless order ν characterizes the regime of frequencies of the solution. The constants A_1 and A_2 are determined from the initial conditions for $u(t)$ (see (35)),

$$A_1 = \left(\nu - \frac{\gamma_0}{2\zeta} - \frac{1}{2}\right)\mathcal{K}_\nu\left(\frac{\gamma_0}{2\zeta}\right) + \frac{\gamma_0}{2\zeta}\mathcal{K}_{\nu-1}\left(\frac{\gamma_0}{2\zeta}\right) \tag{106}$$

$$A_2 = -\left(\nu - \frac{\gamma_0}{2\zeta} - \frac{1}{2}\right)\mathcal{I}_\nu\left(\frac{\gamma_0}{2\zeta}\right) + \frac{\gamma_0}{2\zeta}\mathcal{I}_{\nu-1}\left(\frac{\gamma_0}{2\zeta}\right). \tag{107}$$

Now, substituting the results in (105), we obtain

$$\begin{aligned} u(t) = e^{-\zeta t/2}\Bigg\{&\left[\left(\nu - \frac{\gamma_0}{2\zeta} - \frac{1}{2}\right)\mathcal{K}_\nu\left(\frac{\gamma_0}{2\zeta}\right) + \frac{\gamma_0}{2\zeta}\mathcal{K}_{\nu-1}\left(\frac{\gamma_0}{2\zeta}\right)\right]\mathcal{I}_\nu\left(\frac{\gamma_0}{2\zeta}\,e^{-\zeta t}\right) \\ &-\left[\left(\nu - \frac{\gamma_0}{2\zeta} - \frac{1}{2}\right)\mathcal{I}_\nu\left(\frac{\gamma_0}{2\zeta}\right) - \frac{\gamma_0}{2\zeta}\mathcal{I}_{\nu-1}\left(\frac{\gamma_0}{2\zeta}\right)\right]\mathcal{K}_\nu\left(\frac{\gamma_0}{2\zeta}\,e^{-\zeta t}\right)\Bigg\}. \end{aligned} \tag{108}$$

This same procedure can be repeated for the coefficient $v(t)$, giving the solution

$$v(t) = \frac{e^{-\zeta t/2}}{m_0\zeta}\left[\mathcal{I}_\nu\left(\frac{\gamma_0}{2\zeta}\right)\mathcal{K}_\nu\left(\frac{\gamma_0}{2\zeta}\,e^{-\zeta t}\right) - \mathcal{K}_\nu\left(\frac{\gamma_0}{2\zeta}\right)\mathcal{I}_\nu\left(\frac{\gamma_0}{2\zeta}\,e^{-\zeta t}\right)\right]. \tag{109}$$

The solution of the differential equation for the coefficient $w(t)$ is somewhat different, since this equation is non-homogeneous,

$$\frac{d^2 w}{dt^2} + \zeta\frac{dw}{dt} + \left(\omega_0^2 - \frac{\gamma_0^2}{4}\,e^{-2\zeta t}\right)w = \frac{F(t)}{m_0}\exp\left[\frac{\gamma(t)}{2} - \zeta t\right]. \tag{110}$$

Using the same procedure as before, we obtain

$$\frac{d^2\tilde{w}}{dx^2} + \frac{1}{x}\frac{d\tilde{w}}{dx} - \left(1 + \frac{\nu^2}{x^2}\right)\tilde{w} = Q(x) \tag{111}$$

whose solution is

$$\tilde{w}(x) = B_1\mathcal{I}_\nu(x) + B_2\mathcal{K}_\nu(x) + \int^x [\mathcal{I}_\nu(x)\mathcal{K}_\nu(\tilde{x}) - \mathcal{K}_\nu(x)\mathcal{I}_\nu(\tilde{x})]\tilde{x}Q(\tilde{x})\,d\tilde{x}. \tag{112}$$

With the initial conditions for $w(t)$ given by (37), we obtain

$$\begin{aligned} w(t) = \frac{e^{-\zeta t/2}}{m_0\zeta}\int_0^t \exp\left\{\frac{1}{2}[\gamma(t_1) - \zeta t_1]\right\}&\left[\mathcal{I}_\nu\left(\frac{\gamma_0}{2\zeta}\,e^{-\zeta t}\right)\mathcal{K}_\nu\left(\frac{\gamma_0}{2\zeta}\,e^{-\zeta t_1}\right)\right.\\ &\left.-\mathcal{K}_\nu\left(\frac{\gamma_0}{2\zeta}\,e^{-\zeta t}\right)\mathcal{I}_\nu\left(\frac{\gamma_0}{2\zeta}\,e^{-\zeta t_1}\right)\right]F(t_1)\,dt_1. \end{aligned} \tag{113}$$

Finally, having the set of solutions u, v, w that determines the operator $\boldsymbol{Q}_{\mathrm{H}}(t)$, then by using (42)–(44), it becomes trivial to obtain the set of coefficients μ, κ, η, which determines the momentum operator $\boldsymbol{P}_{\mathrm{H}}(t)$,

$$\begin{aligned} \mu(t) = m_0\zeta\,e^{\zeta t/2}&\left\{\left[\left(\nu - \frac{\gamma_0}{2\zeta} - \frac{1}{2}\right)\mathcal{K}_\nu\left(\frac{\gamma_0}{2\zeta}\right) + \frac{\gamma_0}{2\zeta}\mathcal{K}_{\nu-1}\left(\frac{\gamma_0}{2\zeta}\right)\right]\right.\\ &\times\left[\left(\nu - \frac{\gamma_0}{2\zeta}\,e^{-\zeta t} - \frac{1}{2}\right)\mathcal{I}_\nu\left(\frac{\gamma_0}{2\zeta}\,e^{-\zeta t}\right) - \frac{\gamma_0}{2\zeta}\,e^{-\zeta t}\mathcal{I}_{\nu-1}\left(\frac{\gamma_0}{2\zeta}\,e^{-\zeta t}\right)\right]\\ &-\left[\left(\nu - \frac{\gamma_0}{2\zeta} - \frac{1}{2}\right)\mathcal{I}_\nu\left(\frac{\gamma_0}{2\zeta}\right) - \frac{\gamma_0}{2\zeta}\mathcal{I}_{\nu-1}\left(\frac{\gamma_0}{2\zeta}\right)\right]\\ &\left.\times\left[\left(\nu - \frac{\gamma_0}{2\zeta}\,e^{-\zeta t} - \frac{1}{2}\right)\mathcal{K}_\nu\left(\frac{\gamma_0}{2\zeta}\,e^{-\zeta t}\right) + \frac{\gamma_0}{2\zeta}\,e^{-\zeta t}\mathcal{K}_{\nu-1}\left(\frac{\gamma_0}{2\zeta}\,e^{-\zeta t}\right)\right]\right\} \end{aligned} \tag{114}$$

$$\begin{aligned} \kappa(t) = e^{\zeta t/2}&\left\{\left(\nu - \frac{\gamma_0}{2\zeta}\,e^{-\zeta t} - \frac{1}{2}\right)\left[\mathcal{I}_\nu\left(\frac{\gamma_0}{2\zeta}\right)\mathcal{K}_\nu\left(\frac{\gamma_0}{2\zeta}\,e^{-\zeta t}\right) - \mathcal{K}_\nu\left(\frac{\gamma_0}{2\zeta}\right)\mathcal{I}_\nu\left(\frac{\gamma_0}{2\zeta}\,e^{-\zeta t}\right)\right]\right.\\ &\left.+\frac{\gamma_0}{2\zeta}\,e^{-\zeta t}\left[\mathcal{I}_\nu\left(\frac{\gamma_0}{2\zeta}\right)\mathcal{K}_{\nu-1}\left(\frac{\gamma_0}{2\zeta}\,e^{-\zeta t}\right) + \mathcal{K}_\nu\left(\frac{\gamma_0}{2\zeta}\right)\mathcal{I}_{\nu-1}\left(\frac{\gamma_0}{2\zeta}\,e^{-\zeta t}\right)\right]\right\} \end{aligned} \tag{115}$$

$$\begin{aligned} \eta(t) = e^{\zeta t/2}\int_0^t \exp\left\{\frac{1}{2}[\gamma(t_1) - \zeta t_1]\right\}F(t_1)&\left\{\left(\nu - \frac{\gamma_0}{2\zeta}\,e^{-\zeta t} - \frac{1}{2}\right)\right.\\ &\times\left[\mathcal{I}_\nu\left(\frac{\gamma_0}{2\zeta}\,e^{-\zeta t}\right)\mathcal{K}_\nu\left(\frac{\gamma_0}{2\zeta}\,e^{-\zeta t_1}\right) - \mathcal{K}_\nu\left(\frac{\gamma_0}{2\zeta}\,e^{-\zeta t}\right)\mathcal{I}_\nu\left(\frac{\gamma_0}{2\zeta}\,e^{-\zeta t_1}\right)\right] - \frac{\gamma_0}{2\zeta}\,e^{-\zeta t}\\ &\left.\times\left[\mathcal{I}_{\nu-1}\left(\frac{\gamma_0}{2\zeta}\,e^{-\zeta t}\right)\mathcal{K}_\nu\left(\frac{\gamma_0}{2\zeta}\,e^{-\zeta t_1}\right) + \mathcal{K}_{\nu-1}\left(\frac{\gamma_0}{2\zeta}\,e^{-\zeta t}\right)\mathcal{I}_\nu\left(\frac{\gamma_0}{2\zeta}\,e^{-\zeta t_1}\right)\right]\right\}dt_1. \end{aligned} \tag{116}$$

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